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On the (co)girth of a connected matroid☆

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Abstract

This article studies the girth and cogirth problems for a connected matroid. The problem of finding the cogirth of a graphic matroid has been intensively studied, but studies on the equivalent problem for a vector matroid or a general matroid have been rarely reported. Based on the duality and connectivity of a matroid, we prove properties associated with the girth and cogirth of a matroid whose contraction or restriction is disconnected. Then, we devise algorithms that find the cogirth of a matroid *M* from the matroids associated with the direct sum components of the restriction of *M*. As a result, the problem of finding the (co)girth of a matroid can be decomposed into a set of smaller sub-problems, which helps alleviate the computation. Finally, we implement and demonstrate the application of our algorithms to vector matroids.

Keywords: Matroids; Matroid connectivity; Girth; Cogirth

1. Introduction

Matroid theory started from the algebraic theory of linear dependence and has been employed in many areas such as graph theory, lattice theory, electrical system theory, scheduling, and linear programming. The abstract properties in matroid theory enables various problems, which are equivalent in a matroid framework, to be solved by general matroid-based algorithms. Also, matroid theory suggests profound framework that further enhance and integrate existing specialized algorithms. In our research, we utilize matroid theory to devise algorithms for finding the cogirth of a vector matroid. The (co)girth have practical implication in many applications: for example, the degree of sensor redundancy in a sensor network [13] can be represented by the cogirth of a vector matroid; a key quantity *M* in a robust regression estimator (called *D*-estimator in their paper) studied by Mili and Coakley [8] is simply another form of the cogirth of a vector matroid specified by the transposed data matrix.

We mostly follow Oxley [11] in matroid terminology. For thorough introduction on matroid theory, please see [11,6]. Let *M* be a matroid $(E, \mathscr{I})$ consisting of a ground set *E* and a collection $\mathscr{I}$ of independent subsets of *E*. *M* satisfies so-called *independence augmentation axiom*; i.e., if I_1 and I_2 are in $\mathscr{I}$ and $|I_1| < |I_2|$, then there is an element *e* of $I_2 - I_1$ such that $I_1 \cup e \in \mathscr{I}$. One example of a matroid is a *vector matroid*, $M[\mathbf{A}]$, which is obtained from a matrix **A**, where *E* is the set of column label of **A**, and $\mathscr{I}$ is the set of subsets of *E* such that the subsets are linearly independent. Suppose $X \subseteq E$; then, the rank of *X*, $r(X) = \max\{|I| : I$ is an indepedent subset of $X\}$. A *base* is a maximal independent set, and

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by $\mathscr{B}(M)$ we denote the collection of bases of M. The *dual* matroid M^* has the bases $\mathscr{B}(M^*) = \{E - B : B \in \mathscr{B}(M)\}$. A *circuit* is a minimal dependent set, and by $\mathscr{C}(M)$ we denote the collection of circuits of M. We call $C \in \mathscr{C}(M^*)$ as a *cocircuit*. $\mathscr{B}(M^*)$ and $\mathscr{C}(M^*)$ are also denoted by $\mathscr{B}^*(M)$ and $\mathscr{C}^*(M)$, respectively. The *girth* of a matroid M, $g(M)$ is the cardinality of the smallest circuit in M and the *cogirth* $g^*(M)$ is the cardinality of the smallest cocircuit in M.

Denote a graph by $G = (V, E)$, where $V(G)$ is a non-empty set of vertices and $E(G)$ is a multiset of edges. The cocircuits of a graphic matroid $M(G)$ are the minimal subsets of $E(G)$ whose removal increases the number of components of G, and are also called *cut-sets* of G. Suppose G is connected, the cogirth of $M(G)$ is the *edge-connectivity* of G [10]. The edge-connectivity received attentions from many researchers because it is used to measure how reliable a graph is. For example, Matula [7] developed an algorithm that finds the edge-connectivity and a minimum cut in $O(|V||E|)$ time; Nagamochi and Ibaraki [9] have developed an alternative algorithm to find the edge-connectivity in $O(|E| + \lambda(G)|V|^2)$ time, which is at least as good as the $O(|V||E|)$ time bound, where $\lambda(G)$ is the edge-connectivity.

Unfortunately, the (co)girth problem for a more general matroid (e.g. a vector matroid) is a challenging problem. In 1971, Welsh [15] called for an efficient algorithm for finding the shortest circuit of a vector matroid over a field $\mathbb{F}$, but no polynomial-time algorithm has been found. Vardy [14] later proved that the minimum distance problem in binary coding, which can be converted to the girth problem of a vector matroid over a field $\mathbb{GF}(2)$, is NP-complete. The NP-completeness of the minimum distance problem implies that a polynomial-time algorithm for the (co)girth of a vector matroid as well as of a general matroid unlikely exists [14].

Probably because of this theoretical difficulty, there are not many existing methods to find the (co)girth of a vector matroid; for instance, Mili and Coakley [8] did not specify how to find their cogirth-equivalent quantity in their *D*-estimator. To the best of our knowledge, two existing alternatives for finding the (co)girth are an exhaustive rank testing procedure [13] and the circuit enumeration algorithm. The exhaustive rank testing that finds the cogirth of a vector matroid can be summarized as Algorithm 1. Throughout the paper, denote by $\mathbf{A}_{(-d)}$ the reduced matrix after deleting its d columns and by $r(\mathbf{A})$ the rank of $\mathbf{A}$.

Algorithm 1. *Computing the cogirth of a vector matroid* $M[\mathbf{A}]$
Input: *Matrix* $\mathbf{A} \in \mathbb{R}^{p\times m}$

(1) $d = 1$.
(2) *If there exists* $\mathbf{A}_{(-d)}$ *such that* $r(\mathbf{A}_{(-d)}) < r(\mathbf{A})$, *stop, and the cogirth of* $M[\mathbf{A}]$ *is* d.
(3) $d = d + 1$, *and return to step* (2).

The limitation of this exhaustive rank testing algorithm is that it may run into heavy computation when $g^*(M)$ is large because step (2) in the above procedure takes $\binom{m}{d}$ iterations, and as such, the computation time is proportional to $\sum_{d=1}^{g^*(M)} \binom{m}{d}$.

The circuit enumeration [5] or hyperplane generation [12] algorithms, which are proven to be polynomial-rate enumeration algorithms, provide an alternative way to find the (co)girth. However, in large-size systems, these enumeration algorithms may not be practical, especially when the number of (co)circuits is relatively large. In fact, an enumeration of all circuits (or hyperplanes) is unnecessary if one only wants to find the (co)girth.

In this article, we study the properties and prove theorems regarding the cogirth of a connected matroid. These theorems allow us to enumerate only subsets of $\mathscr{C}^*(M)$ when attempting to find the cogirth. We further devise algorithms based on the theorems. Our algorithms are implemented for vector matroids, and outperform the existing alternatives.

The remainder of this paper is organized as follows. In Section 2, we explain the connectivity of matroids. In Section 3, we present and prove the theorems regarding the (co)girth using the duality and connectivity of a matroid. We present the algorithms for finding the cogirth of a vector matroids, as well as the illustrative examples in Section 4. Finally, we conclude our paper in Section 5.

2. Matroid connectivity

Before we present and prove properties about the (co)girth of a connected matroid in Section 3, we introduce the concept of matroid connectivity. Matroid connectivity has been defined in [11] using matroid restriction and deletion. In order to facilitate the later derivations and discussions, we briefly review the concepts regarding restriction and deletion in Section 2.1 and connectivity of a matroid in Section 2.2. The notations in this section also follow those in [11].

2.1. Restriction and deletion

Let M be the matroid $(E, \mathscr{I})$ consisting of a ground set E and a collection $\mathscr{I}$ of independent subsets of E. Suppose that $X \subseteq E$ and $I \in \mathscr{I}$, Let $\mathscr{I}|X$ be $\{I \subseteq X : I \in \mathscr{I}\}$. Then $(X, \mathscr{I}|X)$ is a matroid called the *restriction of M to X* or the *deletion of E − X from M*. It is denoted by $M|X$ or $M\backslash(E - X)$.

Suppose $S \subseteq E(G)$, then $G\backslash S$ denotes the graph obtained from G by deleting the edges in S. We can easily see that

$$M(G\backslash S) = M(G)\backslash S.$$

Let $\mathbf{A}$ be a matrix, and S be a subset of E, the set of column labels of $\mathbf{A}$. Let $\mathbf{A}\backslash S$ be the matrix obtained from $\mathbf{A}$ by deleting all the columns whose labels are in S. The following property can be found in [11, p. 112].

$$M[\mathbf{A}]\backslash S = M[\mathbf{A}\backslash S].$$

2.2. Connectivity of a matroid

The matroid M is disconnected if and only if, for some proper non-empty subset T of $E(M)$,

$$\mathscr{I}(M) = \{I_1 \cup I_2 : I_1 \in \mathscr{I}(M|T), I_2 \in \mathscr{I}(M|(E - T))\}.$$

This property of a disconnected matroid implies that

$$r(M) = r(T) + r(E - T).$$

Since $r^*(T) = |T| - r(M) + r(E - T)$, where $r^*(T) = r((M|T)^*)$,

$$r(T) + r^*(T) - |T| = 0.$$

Naturally, this implies that connectivity is self-dual. Hence, M is connected if and only if M^* is connected.

Let $M_1, M_2, \ldots, M_n$ be matroids on disjoint ground sets $E_1, E_2, \ldots, E_n$, respectively. Also let $\mathscr{I}_1, \mathscr{I}_2, \ldots, \mathscr{I}_n$ be the collections of independent sets in $M_1, M_2, \ldots, M_n$, respectively. Suppose $E = E_1 \cup E_2 \cup \cdots \cup E_n$ and $\mathscr{I} = \{I_1 \cup I_2 \cup \cdots \cup I_n : I_j \in \mathscr{I}_j$ for all $j \in \{1, 2, \ldots, n\}\}$, then $M = (E, \mathscr{I})$ is a matroid and denoted by $M_1 \oplus M_2 \oplus \cdots \oplus M_n$. We call $M_1, M_2, \ldots, M_n$ the *direct sum components of M*, and M is called the *direct sum* or 1-*sum* of $M_1, M_2, \ldots, M_n$,

The direct sum has following properties which are straightforward to prove.

- $\mathscr{C}(M_1 \oplus M_2 \oplus \cdots \oplus M_n) = \mathscr{C}(M_1) \cup \mathscr{C}(M_2) \cup \cdots \cup \mathscr{C}(M_n)$,
- $(M_1 \oplus M_2 \oplus \cdots \oplus M_n)^* = M_1^* \oplus M_2^* \oplus \cdots \oplus M_n^*$.

From these two properties, we can conclude that

$$\mathscr{C}^*(M_1 \oplus M_2 \oplus \cdots \oplus M_n) = \mathscr{C}^*(M_1) \cup \mathscr{C}^*(M_2) \cup \cdots \cup \mathscr{C}^*(M_n). \quad (1)$$

Hence, (co)girth of a disconnected matroid is the minimum of the (co)girths of its direct sum components.

Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be matrices over field $\mathbb{F}$, and $M_1 = M[\mathbf{A}_1]$ and $M_2 = M[\mathbf{A}_2]$. The direct sum $M = M_1 \oplus M_2$ can be regarded as a matroid on a matrix $\mathbf{A}$ over field $\mathbb{F}$, such that

$$\mathbf{A} = \left(\begin{array}{c|c} \mathbf{A}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{A}_2 \end{array}\right). \quad (2)$$

From Eq. (1), we can find the (co)girth of $M[\mathbf{A}]$ from the minimum of those of M_1 and M_2. The matrix $\mathbf{A}$ is said to be in the *block diagonal form* due to its shape.

3. (Co)girth of a connected matroid

3.1. The cogirth problem

Let M be a connected matroid. Suppose there exists $S \subset E(M)$ such that $M \backslash S$ is disconnected and $M_1, M_2, \ldots, M_n$ are the direct sum components of $M \backslash S$. Then, $M \backslash S = M_1 \oplus M_2 \oplus \cdots \oplus M_n$.

For $i = 1, \ldots, n$, we define

- $\mathscr{C}_i^*(M) = \{D \in \mathscr{C}^*(M) : D \subseteq E(M_i) \cup S\}$, and
- $c_i^*(M) = c_i^* = \min\{|D| : D \in \mathscr{C}_i^*(M)\}$.

Please note that $\mathscr{C}_i^*(M)$ is *not* equivalent to $\mathscr{C}^*(M_i)$.

The subsequent lemmas and theorems characterize the cogirth of a connected matroid. First, we present a result that will be used in the proofs of other lemmas and theorems; its proof is omitted because this is a straightforward result from the independence augmentation axiom in Section 1.

Lemma 1. *Let I be an independent set in a matroid M. There exists a base B containing I in M.*

Lemmas 2 and 3 characterize the cogirth of a matroid when the restriction of the matroid can be decomposed into two direct sum components.

Lemma 2. *Suppose $M \backslash S = M_1 \oplus M_2$. Let D be a cocircuit in M, which is not in $\mathscr{C}_1^*(M)$ or $\mathscr{C}_2^*(M)$. Then,*

$$|D| \geqslant \max(c_1^* - |S|, 1) + \max(c_2^* - |S|, 1).$$

Proof. Let $P_1 = D \cap E(M_1)$, $P_2 = D \cap E(M_2)$, and $T = D \cap S$. Obviously, $D = P_1 \cup P_2 \cup T$. Because D is not in $\mathscr{C}_1^*(M)$ or $\mathscr{C}_2^*(M)$, P_1 and P_2 are not empty sets. First, we will show that $P_1 \cup S$ and $P_2 \cup S$ are codependent sets in M.

Assuming $P_1 \cup S$ is a coindependent set in M, there exists a base $B \subseteq E(M) - (P_1 \cup S) = E(M \backslash S) - P_1$ in M. Since $B \subseteq E(M \backslash S)$, B is also a base of $M \backslash S = M_1 \oplus M_2$. Then, there exists a base B_1 in M_1, and a base B_2 in M_2 such that $B = B_1 \cup B_2$. As D is a cocircuit and P_1 is not an empty set (D is not in $\mathscr{C}_2^*$), $P_2 \cup T$ is a coindependent set. Therefore, $r(E(M) - (P_2 \cup T)) = r(M)$. Since B_1 is an independent set in M_1, B_1 is also an independent set in $M \backslash (P_2 \cup T)$. By Lemma 1, there is a base B' in $M \backslash (P_2 \cup T)$ such that $B_1 \subseteq B'$. Since $r(M) = r(M \backslash (P_2 \cup T))$, B' is also a base in M.

Suppose $P_1 \cap B' \neq \emptyset$. Then, there exists $e \in P_1 \cap B'$. Since $B_1 \subseteq E(M_1) - P_1$, e is not in B_1. However, $e \in E(M_1)$, and B_1 is a base in M_1. Therefore, $\{e\} \cup B_1$ should be a dependent set in M_1. This contradicts the assumption that B' is a base in M, and thus, $P_1 \cap B' = \emptyset$. This implies that $B' \subseteq E - (P_1 \cup P_2 \cup T) = E - D$, which contradicts the assumption that D is a cocircuit. Therefore, $P_1 \cup S$ is a codependent set. Likewise, we can prove $P_2 \cup S$ is a codependent set.

Because $P_1 \cup S$ is a codependent set, $|P_1 \cup S| \geqslant c_1^*$, or

$$|P_1| \geqslant c_1^* - |S|.$$

Since $|P_1| \geqslant 1$,

$$|P_1| \geqslant \max(c_1^* - |S|, 1).$$

Likewise, it can be proven that $|P_2| \geqslant \max(c_2^* - |S|, 1)$. Therefore,

$$|P_1| + |P_2| \geqslant \max(c_1^* - |S|, 1) + \max(c_2^* - |S|, 1).$$

Thus, $|D| = |P_1 \cup P_2 \cup T| \geqslant \max(c_1^* - |S|, 1) + \max(c_2^* - |S|, 1)$. □

Lemma 3. *If $M \backslash S = M_1 \oplus M_2$, and $g^*(M) \geqslant 2|S| - 1$, then*

$$g^*(M) = \min(c_1^*, c_2^*).$$

Proof. Let D be a cocircuit in M, which is not in $\mathscr{C}_1^*(M)$ or $\mathscr{C}_2^*(M)$. What we need to prove is

$$|D| \geqslant \min(c_1^*, c_2^*). \tag{3}$$

From Lemma 2,

$$|D| \geqslant \max(c_1^* - |S|, 1) + \max(c_2^* - |S|, 1).$$

Since $\max(c_1^* - |S|, 1) + \max(c_2^* - |S|, 1) \geqslant c_1^* + c_2^* - 2|S|$,

$$|D| \geqslant c_1^* + c_2^* - 2|S| \geqslant 2\min(c_1^*, c_2^*) - 2|S|.$$

Because $|D| \geqslant 2|S| - 1$,

$$|D| \geqslant 2\min(c_1^*, c_2^*) - |D| - 1,$$

or

$$|D| \geqslant \min(c_1^*, c_2^*) - \tfrac{1}{2}.$$

Therefore,

$$|D| \geqslant \min(c_1^*, c_2^*). \qquad \square$$

Lemma 3 can be extended to general cases with $M_1, M_2, \ldots, M_n$, as stated in Theorem 4, and it can be easily proven using Lemma 3.

Theorem 4. *If* $M \backslash S = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ *and* $g^*(M) \geqslant 2|S| - 1$, *then*

$$g^*(M) = \min_{i \in \{1,2,\ldots,n\}} c_i^*.$$

Theorem 4 tells us that the cogirth of the connected matroid M can be obtained from $c_1^*, c_2^*, \ldots, c_n^*$ once the cogirth is known to be larger than the bound. The bound in the above results is specified as two times the cardinality of S subtracted by one. The following results will further relax this bound.

For $J \subseteq \{1, \ldots, n\}$, we define

- $\mathscr{C}_J^*(M) = \{D \in \mathscr{C}^*(M) : D \subseteq \bigcup_{j \in J} E(M_j) \cup S\}$, and
- $c_J^* = \min\{|D| : D \in \mathscr{C}_J^*\}$.

A *k-subset* is a subset of a set on m elements containing exactly k elements. The number of k-subsets on m elements is $\binom{m}{k}$. Let $\mathscr{J}_k$ be a collection of all the k-subsets of $\{1, 2, \ldots, n\}$.

Lemma 5. *Let* $Y \in \mathscr{J}_k$. *If* $M \backslash S = M_1 \oplus M_2 \oplus \cdots \oplus M_n$, *and* $c_Y^* \geqslant (k/(k-1))|S| - 1$, *then*

$$c_Y^* = \min_{y \in Y} c_{Y-\{y\}}^*.$$

Proof. Let $D \in \mathscr{C}_Y^*(M)$, $P_i = D \cap E(M_i)$ for $i \in Y$, and $T = D \cap S$.

Our proof is divided into two cases. First, we will prove for the case that there exists $i \in Y$ such that $|P_i| = 0$. Second, we will prove the other case that every P_i for $i \in Y$ is not an empty set.

(1) If there exists i such that P_i is an empty set, then $D \in \mathscr{C}_{Y-\{i\}}^*$. Then,

$$|D| \geqslant c_{Y-\{i\}}^* \geqslant \min_{y \in Y} c_{Y-\{y\}}^*. \tag{4}$$

(2) When all P_i for $i \in Y$ is not an empty set; like the proof for Lemma 3, we can prove that $P_i \cup S$ is a codependent set in M for $i \in Y$. Hence, for $y \in Y$,

$$\sum_{i \in Y-\{y\}} |P_i| \geqslant \sum_{i \in Y-\{y\}} c_i^* - (k-1)|S|.$$

Since $\sum_{i\in Y-\{y\}} c_i^* \geqslant (k-1)c_{Y-\{y\}}^*$,

$$\sum_{i\in Y-\{y\}} |P_i| \geqslant (k-1)c_{Y-\{y\}}^* - (k-1)|S|.$$

Since $\sum_{y\in Y} c_{Y-\{y\}}^* \geqslant k \min_{y\in Y} c_{Y-\{y\}}^*$, adding up for all $y \in Y$,

$$\sum_{y\in Y}\sum_{i\in Y-\{y\}} |P_i| \geqslant k(k-1)\min_{y\in Y} c_{Y-\{y\}}^* - k(k-1)|S|.$$

Since $\sum_{y\in Y}\sum_{i\in Y-\{y\}} |P_i| = (k-1)\sum_{i\in Y}|P_i| = (k-1)|D| - (k-1)|T|$,

$$|D| \geqslant k\min_{y\in Y} c_{Y-\{y\}}^* - k|S| + |T| \geqslant k\min_{y\in Y} c_{Y-\{y\}}^* - k|S|.$$

Because $k|S| \leqslant (k-1)|D| + (k-1)$,

$$|D| \geqslant k\min_{y\in Y} c_{Y-\{y\}}^* - (k-1)|D| - (k-1),$$

or

$$|D| \geqslant \min_{y\in Y} c_{Y-\{y\}}^* - \frac{(k-1)}{k}.$$

Since $c_Y^* = \min\{|D| : D \in \mathscr{C}_Y^*\}$,

$$c_Y^* \geqslant \min_{y\in Y} c_{Y-\{y\}}^*. \tag{5}$$

From Eqs. (4) and (5),

$$c_Y^* \geqslant \min_{y\in Y} c_{Y-\{y\}}^*.$$

Since $\mathscr{C}_Y^*$ contains all the cocircuits in $\mathscr{C}_{Y-\{y\}}^*$ for all $y \in Y$,

$$c_Y^* = \min_{y\in Y} c_{Y-\{y\}}^*. \qquad \square$$

Using Lemma 5, we prove Theorem 6, which generalizes Theorem 4. In fact, Theorem 4 is the special case of Theorem 6 when $k = 1$.

Theorem 6. *If $M\backslash S = M_1 \oplus M_2 \oplus \cdots \oplus M_n$, and $g^*(M) \geqslant ((k+1)/k)|S| - 1$, then*

$$g^*(M) = \min_{J\in\mathscr{J}_k} c_J^*.$$

Proof. Suppose $g^*(M) = c_{\{1,2,\ldots,n\}}^* \neq \min_{J\in\mathscr{J}_k} c_J^*$. Then, there exists k' such that $k+1 \leqslant k' \leqslant n$, and

$$\min_{Y\in\mathscr{J}_{k'}} c_Y^* \neq \min_{X\in\mathscr{J}_{k'-1}} c_X^*.$$

Since $k' \geqslant k+1$,

$$\min_{Y\in\mathscr{J}_{k'}} c_Y^* \geqslant g^*(M) \geqslant \frac{k+1}{k}|S| - 1 \geqslant \frac{k'}{k'-1}|S| - 1.$$

The condition in Lemma 5 is satisfied, so

$$\min_{Y\in\mathscr{J}_{k'}} c_Y^* = \min_{Y\in\mathscr{J}_{k'}}\left\{\min_{y\in Y} c_{Y-\{y\}}^*\right\} = \min_{X\in\mathscr{J}_{k'-1}} c_X^*,$$

which leaves us with a contradiction. Thus,

$$g^*(M) = c^*_{\{1,2,\ldots,n\}} = \min_{J\in\mathscr{J}_k} c^*_J. \qquad \square$$

Theorems 4 and 6 provide the basis for our latter algorithms, which allows us to find the cogirth without having to enumerate all cocircuits when a certain bound condition is satisfied. The details of applying Theorems 4 and 6 will be explained in Section 4.

3.2. The girth problem

In this section, we introduce the operation of *contraction* as the dual of deletion in order to solve the girth problem of a matroid. Let M be a matroid on E, and T be a subset of E. The *contraction of T from M*, M/T is given by

$$M/T = (M^*\backslash T)^*,$$

and also called *contraction of M to* $E - T$.

Let G be a graph and T be a subset of $E(G)$. Oxley proved [11, Proposition 3.2.1] that

$$M(G)/T = M(G/T), \tag{6}$$

where G/T denotes the graph obtained from G by contracting the edges in T. Please note that G/T in (6) may not necessarily be a simple graph but could be a multigraph with loops and/or parallel edges, and the contraction in (6) allows to contract cycles in a graphic matroid, which is different from usual graph contraction. In fact, $M(G)/T$ generalizes the contraction for graphs; for more details, please refer to Sections 3.1 and 3.2 in [11].

Finding the contraction of a vector matroid, however, is more involved than that of a graphic matroid. Oxley [11] presented a contraction of a vector matroid by one column of the matrix as follows: suppose e is the label of a non-zero column of $\mathbf{A}$. Let $\mathbf{A}'$ be a transformed matrix from $\mathbf{A}$ by pivoting e to be a unit vector.

$$M[\mathbf{A}]/e = M[\mathbf{A}']/e = M[\mathbf{A}'/e],$$

where $\mathbf{A}'/e$ denotes the matrix obtained from $\mathbf{A}'$ by deleting the row and column containing the unique non-zero entry in e.

Since the girth of M is the cogirth of M^*, we can utilize Theorems 4 and 6 to solve the girth problem. According to duality between M and M^*, the contraction of M is disconnected if and only if the restriction of M^* is disconnected. Suppose there exists $T \subset E$ such that M/T is disconnected and $M_1, M_2, \ldots, M_n$ are the direct sum components of M/T. Then,

$$(M/T)^* = M^*\backslash T = M_1^* \oplus M_2^* \oplus \cdots \oplus M_n^*.$$

For $i = 1, \ldots, n$, we define

- $\mathscr{C}_i(M) = \{C \in \mathscr{C}(M) : C \subseteq E(M_i) \cup T\}$, and
- $c_i = \min\{|C| : C \in \mathscr{C}_i(M)\}$.

Note that $g(M)$ is the girth of M. Similar to Theorem 4, we can derive the following theorem on the girth of a connected matroid.

Theorem 7. *If* $M/T = M_1 \oplus M_2 \oplus \cdots \oplus M_n$, *and* $g(M) \geqslant 2|T| - 1$, *then*

$$g(M) = \min_{i\in\{1,2,\ldots,n\}} c_i.$$

Proof. Since $M^*\backslash T = M_1^* \oplus M_2^* \oplus \cdots \oplus M_n^*$, by Theorem 4

$$g^*(M^*) = \min_{i\in\{1,2,\ldots,n\}} c_i^*(M_i^*).$$

By the duality, $g^*(M^*) = g(M)$, and $c_i^*(M^*) = c_i(M)$. $\square$

Again, the bound $2|T| - 1$ in the above theorem can be relaxed.
For $J \subseteq \{1, \dots, n\}$,

- $\mathscr{C}_J(M) = \{C \in \mathscr{C}(M) : C \subseteq \bigcup_{j \in J} E(M_j) \cup T\}$, and
- $c_J = \min\{|C| : C \in \mathscr{C}_J\}$.

Again, let $\mathscr{J}_k$ be a collection of all the k-subsets of $\{1, 2, \dots, n\}$.

Theorem 8. *If* $M/T = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ *and* $g(M) \geqslant ((k+1)/k)|T| - 1$, *then*

$$g(M) = \min_{J \in \mathscr{J}_k} c_J.$$

The proof is omitted.

4. Algorithms for vector matroids

In this section, we present the algorithms for finding the cogirth of a vector matroid based on Theorems 4 and 6. The basic idea in utilizing Theorems 4 and 6 is to decompose the matrix on which a vector matroid is specified. In so doing, Section 4.1 presents an n-way hypergraph partitioning procedure to find S such that $M \backslash S$ can be disconnected and to find $E(M_i)$, the ground set of the ith direct sum component for $i = 1, 2, \dots, n$; Section 4.2 has the details of our algorithms; and Section 4.3 discuss how these algorithms for solving the cogirth problem can be applied to the girth problem using the standard representative matrices.

4.1. Bordered-block structure

Let $\mathbf{A}$ be a $p \times m$ matrix over the field $\mathbb{F}$ and $M[\mathbf{A}]$ be the vector matroid specified by $\mathbf{A}$. In order to utilize Theorems 4 and 6, we need to permute $\mathbf{A}$ into a *bordered block diagonal form* (BBDF) such as

$$\mathbf{A}_{\mathrm{BBDF}} = \begin{pmatrix} \mathbf{A}_1 & & & & \mathbf{S}_1 \\ & \mathbf{A}_2 & & & \mathbf{S}_2 \\ & & \ddots & & \vdots \\ & & & \mathbf{A}_n & \mathbf{S}_n \end{pmatrix},$$

where $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ are called the *blocks*, and a submatrix $(\mathbf{S}_1^{\mathrm{T}}\ \mathbf{S}_2^{\mathrm{T}}\ \cdots\ \mathbf{S}_n^{\mathrm{T}})^{\mathrm{T}}$ is called the *border*. In fact, the block diagonal form in Eq. (2) is a special case of this BBDF where there is no border in it.

Denote by $\mathbf{A}[I, J]$ the submatrix of $\mathbf{A}$ with the row set I and the column set J, namely, $\mathbf{A}[I, J] = (a_{ij} | i \in I, j \in J)$; and also let $R = \mathrm{Row}(\mathbf{A})$ and $E = \mathrm{Col}(\mathbf{A})$. Denote by S the set of column labels corresponding to the border, and we call S a *separating set*. As such, the notation $\mathbf{A}[R, E - S]$ represents the rest of the original $\mathbf{A}$ matrix after a separating set S is removed. Let $(R_1, R_2, \dots, R_n)$ and $(E_1, E_2, \dots, E_n)$ be the partition of R and $E - S$ such that R_i and E_i are the row set and the column set of the ith block $\mathbf{A}_i$, i.e., $\mathbf{A}_i = \mathbf{A}[R_i, E_i]$. Since $M[\mathbf{A}[R, E - S]] = M[\mathbf{A}] \backslash S$, we can easily derive the following lemma.

Lemma 9. *If* S *is a separating set and* $\mathbf{A}[R_1, E_1], \mathbf{A}[R_2, E_2], \dots, \mathbf{A}[R_n, E_n]$ *are the blocks of* $\mathbf{A}[R, E - S]$, *then*

$$M[\mathbf{A}] \backslash S = M[\mathbf{A}[R_1, E_1]] \oplus M[\mathbf{A}[R_2, E_2]] \oplus \cdots \oplus M[\mathbf{A}[R_n, E_n]].$$

There are a few research papers reporting methods on permuting a matrix $\mathbf{A}$ into the BBDF. Ferris and Horn [3] proposed a two phase approach to find a bordered block structure. Ayknant et al. [1] used hypergraph models to change the permutation problem to an n-way hypergraph partitioning problem. Tools such as hMeTis [4] and PaToH [2] provide stable results very quickly in solving the n-way hypergraph partitioning problems.

A hypergraph $H = (V, N)$ is defined as a set of vertices V and a multiset of hyperedges N, which are a subset of V. Let $\mathbf{A}$ be a $p \times m$ matrix. *Hypergraph representation* of $\mathbf{A}$ is a $H = (V, N)$ such that V is a set of row labels of $\mathbf{A}$

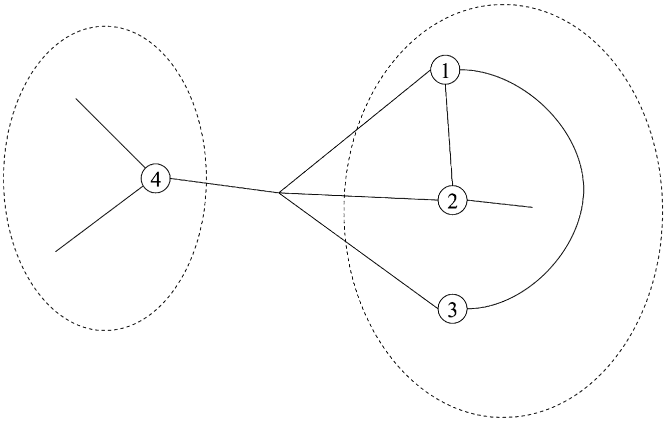

Fig. 1. Hypergraph representation.

and $N_i \in N$ contains the vertices corresponding to the rows that have a non-zero entry in column i. Let $i \in V$. Then, $i \in N_j$ if and only if a_{ij} is non-zero, where a_{ij} is the (i, j)-entry of $\mathbf{A}$.

For example,

$$\mathbf{A} = \begin{pmatrix} 0 & a_{12} & a_{13} & 0 & 0 & a_{16} \\ 0 & a_{22} & 0 & 0 & a_{25} & a_{26} \\ 0 & 0 & a_{33} & 0 & 0 & a_{36} \\ a_{41} & 0 & 0 & a_{44} & 0 & a_{46} \end{pmatrix}.$$

The hypergraph representation of $\mathbf{A}$ is $H = (V, N)$ such that $V = \{1, 2, 3, 4\}$ and $N = \{\{4\}, \{1, 2\}, \{1, 3\}, \{4\}, \{2\}, \{1, 2, 3, 4\}\}$. Fig. 1 illustrates H. In Fig. 1, the circled numbers (①, ②, ③, and ④) represent the vertices in V, and solid lines represent the hyperedges. For example, the rightmost hyperedge connecting ① and ③ represents $\{1, 3\}$, which corresponds to the third column in $\mathbf{A}$. Dashed ellipses show resulting partitions after removing hyperedge $\{1, 2, 3, 4\}$.

The n-way hypergraph partitioning problem is to find a set of hyperedges of a minimum size whose removal disconnects the n vertex parts of a hypergraph. Using one of the aforementioned n-way hypergraph partitioning tools, we can see that the hypergraph H is disconnected by deleting the hyperedge $\{1, 2, 3, 4\}$, and the resulting partitions are $\{1, 2, 3\}$ and $\{4\}$. In Fig. 1, these two partitions are contained by two dashed ellipses. In fact, the hyperedge $\{1, 2, 3, 4\}$ corresponds to the border of $\mathbf{A}$ and the partitions correspond to the blocks. Then, in the above example of $\mathbf{A}$, the separating set S is $\{6\}$, which corresponds to the hyperedge $\{1, 2, 3, 4\}$. Since we find two partitions, there are two blocks: i.e. $\mathbf{A}[R_1, E_1] = \mathbf{A}[\{1, 2, 3\}, \{2, 3, 5\}]$ and $\mathbf{A}[R_2, E_2] = \mathbf{A}[\{4\}, \{1, 4\}]$.

The aforementioned tools for solving the n-way hypergraph partitioning problem need users to specify the resulting number of blocks, n. The bounds in Theorems 4 and 6 actually provide some guideline in helping select the value of n. For a vector matroid specified on a matrix, the bounds are the switching point when one can start testing smaller submatrices in an attempt to find the cogirth. Therefore, the earlier the switch can be done, the bigger the computation benefit is. In that spirit, we recommend using an n that gives the smallest bound in the corresponding theorems. In the next subsection, two versions of algorithm, Algorithms 2 and 3, are devised based on Theorems 4 and 6, respectively. When using Algorithm 2, the n that yields a BBDF with the smallest $|S|$ should be used; when using Algorithm 3, we recommend using an n so that $(n/(n-1))|S|$ is minimized.

4.2. *Finding cogirth using matroid structure*

Our algorithm aims at finding the cogirth of a matroid specified by a sparse matrix. A sparse matrix usually possesses certain type of structure, which provides an opportunity for us to reduce the computation time when seeking for a high degree of a cogirth. To an extreme, if a matrix can be permuted into a block diagonal form, then $|S| = 0$. Hence, by Theorem 4, $g^*(M)$ is simply the smallest value of the cogirths associated with blocks; this is the case for the matrix in Eq. (2). Obviously, testing the smaller submatrix is usually much less expensive than testing the original matrix. In general, a matrix in the block diagonal form is not very common. A common manifestation of a sparse matrix usually takes the format of the BBDF as illustrated in Section 4.1.

Consider a $p \times m$ matrix $\mathbf{A}$ with n blocks in its BBDF. Lemma 11 shows the relationship between $c_J^*(M)$ and the cogirth of a submatrix, where J is a subset of $\{1, 2, \ldots, n\}$. In order to prove Lemma 11, the following remark is derived using linear algebra.

Remark 10. The cocircuit in $M[\mathbf{A}[\bigcup_{i \in J} R_i, \bigcup_{j \in J} E_j \cup S]]$ is a codependent set in M.

From Remark 10, we can conclude,

$$c_J^*(M[\mathbf{A}]) \leqslant c^* \left(M \left[\mathbf{A} \left[\bigcup_{i \in J} R_i, \bigcup_{j \in J} E_j \cup S \right] \right] \right). \tag{7}$$

Lemma 11 presents that $c_J^*(M)$ can be obtained from the cogirth of a submatrix associated with the blocks and the border when $r(M \backslash S) = r(M)$.

Lemma 11. *If* $r(M \backslash S) = r(M)$,

$$c_J^*(M[\mathbf{A}]) = c^* \left(M \left[\mathbf{A} \left[\bigcup_{i \in J} R_i, \bigcup_{j \in J} E_j \cup S \right] \right] \right),$$

where J is a subset of $\{1, 2, \ldots, n\}$.

Proof. Let $r = r(M)$ and $r_i = r(M_i)$ for $i = 1, 2, \ldots, n$. Since $r(M \backslash S) = r(M)$,

$$r = r_1 + r_2 + \cdots + r_n.$$

Let $D \in \mathscr{C}_J^*(M[\mathbf{A}])$. Then,

$$r(\mathbf{A}[R, E - D]) = r - 1.$$

Since $r(M \backslash S) = r(M)$,

$$r \left(\mathbf{A} \left[R - \bigcup_{i \in J} R_i, E - \left(\bigcup_{j \in J} E_j \cup S \right) \right] \right) = r - \sum_{j \in J} r_j.$$

Hence,

$$r \left(\mathbf{A} \left[\bigcup_{i \in J} R_i, \left(\bigcup_{j \in J} E_j \cup S \right) - D \right] \right) = \sum_{j \in J} r_j - 1.$$

Therefore, the cocircuits in $\mathscr{C}_J^*(M[\mathbf{A}])$ is a codependent set in $M[\mathbf{A}[\bigcup_{i \in J} R_i, \bigcup_{j \in J} E_j \cup S]]$, and

$$c_J^*(M[\mathbf{A}]) \geqslant c^* \left(M \left[\mathbf{A} \left[\bigcup_{i \in J} R_i, \bigcup_{j \in J} E_j \cup S \right] \right] \right).$$

Combining with Eq. (7),

$$c_J^*(M[\mathbf{A}]) = c^* \left(M \left[\mathbf{A} \left[\bigcup_{i \in J} R_i, \bigcup_{j \in J} E_j \cup S \right] \right] \right). \qquad \square$$

As long as $g^*(M)>1$, the bound condition in Theorem 4, $g^*(M)\geqslant 2|S|-1$, implies that $g^*(M)>|S|$. From the definition of cogirth, $g^*(M)>|S|$ means that S is not a cocircuit, and thus, $r(M\backslash S)=r(M)$. This is equivalent to saying that when $g^*(M)>1$ and the bound condition is also satisfied, the condition $r(M\backslash S)=r(M)$ will be satisfied. The following cogirth-finding algorithm is based on Theorem 4 and Lemma 11.

Algorithm 2. *Computing the cogirth of a vector matroid* $M[\mathbf{A}]$.

Input: *Matrix* $\mathbf{A}\in\mathbb{R}^{p\times m}$, *separating set* S, *and row set and column set of blocks* $(R_1, R_2, \ldots, R_n)$ *and* $(E_1, E_2, \ldots, E_n)$.

(1) $d=1$.
(2) *If there exists* $\mathbf{A}_{(-d)}$ *such that* $r(\mathbf{A}_{(-d)})<r(\mathbf{A})$, *stop and* $g^*(M[\mathbf{A}])=d$.
(3) $d=d+1$.
(4) *If* $d<2|S|-1$, *repeat step* (2)–(3).
(5) *For* $i=1,2,\ldots,n$, *if there exists* $\mathbf{A}[R_i, E_i\cup S]_{(-d)}$ *such that* $r(\mathbf{A}[R_i, E_i\cup S]_{(-d)})<r(\mathbf{A}[R_i, E_i\cup S])$, *stop and* $g^*(M[\mathbf{A}])=d$.
(6) $d=d+1$, *and return to step* (5).

As a consequence of Theorem 6, the following corollary is derived to help establish Algorithm 3, which can further save the computation time by reducing the bound $(2|S|-1)$ in the fourth step of Algorithm 2 to $((n/n-1)|S|-1)$.

Corollary 12. *If* $M\backslash S=M_1\oplus M_2\oplus\cdots\oplus M_n$, *and* $g^*(M)\geqslant((k+1)/k)|S|-1$,

$$g^*(M)=\min_{J\in\mathscr{J}_k} c^*\left(M\left[\mathbf{A}\left[\bigcup_{i\in J}R_i, \bigcup_{j\in J}E_j\cup S\right]\right]\right).$$

Recall that $\mathscr{J}_k$ denotes the collection of all the k-subsets of $\{1,2,\ldots,n\}$.

Proof. Suppose $|S|\neq 0$. (The case $|S|=0$ is straightforward, hence omitted.) Since $g^*(M)\geqslant((k+1)/k)|S|-1>|S|-1$,

$$g^*(M)\geqslant|S|.$$

Hence, strict subsets of S are not cocircuits in M.

(1) If S is not a cocircuit, $r(M\backslash S)=r(M)$. By Theorem 6 and Lemma 11,

$$g^*(M)=\min_{J\in\mathscr{J}_k} c_J^*=\min_{J\in\mathscr{J}_k} c^*\left(M\left[\mathbf{A}\left[\bigcup_{i\in J}R_i, \bigcup_{j\in J}E_j\cup S\right]\right]\right). \tag{8}$$

(2) If S is a cocircuit in M, then $g^*(M)=|S|$. Suppose S is coindependent set in $M[\mathbf{A}[R_i, E_i\cup S]]$ for all $i\in\{1,2,\ldots,n\}$. Then,

$$\sum_{i=1}^{n} r(\mathbf{A}[R_i, E_i])=r(\mathbf{A}).$$

This contradicts the assumption that S is a cocircuit in M. Therefore, there exists $i\in\{1,2,\ldots,n\}$ such that S is a codependent set in $M[\mathbf{A}[R_i, E_i\cup S]]$. By Remark 10, for $j\in\{1,2,\ldots,n\}$, the sizes of the cocircuits in $M[\mathbf{A}[R_j, E_j\cup S]]$ are not smaller than $g^*(M)$. Therefore,

$$g^*(M)=\min_{i\in\{1,2,\ldots,n\}} c^*(M[\mathbf{A}[R_i, E_i\cup S]]). \tag{9}$$

From Eqs. (8) and (9), we can conclude,

$$g^*(M)=\min_{J\in\mathscr{J}_k} c_J^*=\min_{J\in\mathscr{J}_k} c^*\left(M\left[\mathbf{A}\left[\bigcup_{i\in J}R_i, \bigcup_{j\in J}E_j\cup S\right]\right]\right). \qquad \square$$

Corollary 12 enables us to establish Algorithm 3, which uses a smaller bound, $((n/(n-1))|S|-1)$, than $(2|S|-1)$ in Algorithm 2. When using the bound of $((n/(n-1))|S|-1)$, instead of testing the rank of a submatrix $\mathbf{A}[R_i, E_i \cup S]$ generated from an individual block, we can test one of the possible submatrices $\mathbf{A}[\bigcup_{j\in J} R_j, \bigcup_{j\in J} E_j \cup S]$ that combine blocks. As specified in Theorem 6 and Corollary 12, J is an element in $\mathscr{J}_k$, which is the collection of all the k-subsets of $\{1, 2, \ldots, n\}$. We therefore need additional steps in the next algorithm to sort through the combinations of blocks for Corollary 12 to be applied.

In so doing, denote by $\mathscr{A}_k$ a collection of matrices such that

$$\mathbf{A}\left[\bigcup_{i\in J} R_i, \bigcup_{j\in J} E_j \cup S\right] \in \mathscr{A}_k \quad \text{for all } J \in \mathscr{J}_k.$$

Obviously, $\mathscr{A}_1$, i.e., $k=1$, is the collection of $\{\mathbf{A}[R_1, E_1 \cup S], \ldots, \mathbf{A}[R_n, E_n \cup S]\}$, and $\mathscr{A}_n$, i.e., $k=n$, contains only $\mathbf{A}$. Denote by $N_{(-d)}(k)$ the total number of reduced matrices generated from *all* the matrices in $\mathscr{A}_k$, so

$$N_{(-d)}(k) = \sum_{J\in\mathscr{J}_k} \binom{\left|\bigcup_{j\in J} E_j\right| + |S|}{d}.$$

The algorithm using Corollary 12 is as follows.

Algorithm 3. *Computing the cogirth of a vector matroid* $M[\mathbf{A}]$

Input: *Matrix* $\mathbf{A} \in \mathbb{R}^{p\times m}$, *separating set S, and row set and column set of blocks* $(R_1, R_2, \ldots, R_n)$ *and* $(E_1, E_2, \ldots, E_n)$.

(1) $d=1$.
(2) *If there exists* $\mathbf{A}_{(-d)}$ *such that* $r(\mathbf{A}_{(-d)}) < r(\mathbf{A})$, *stop and* $g^*(M[\mathbf{A}]) = d$.
(3) $d = d+1$.
(4) *If* $d < (n/(n-1))|S| - 1$, *then repeat step* (2)–(3).
(5) *Find an integer* k^* *such that*

$$k^* = \operatorname*{argmin}_{(|S|/(d-|S|+1)) \leqslant k \leqslant n} N_{(-d)}(k).$$

(6) *For* $\mathbf{A}' \in \mathscr{A}_{k^*}$, *if there exists* $\mathbf{A}'_{(-d)}$ *such that* $r(\mathbf{A}'_{(-d)}) < r(\mathbf{A}')$, *stop and* $g^*(M[\mathbf{A}]) = d$.
(7) $d = d+1$, *and return to step* (5).

At the fifth step, the algorithm selects k that requires the least computation time. The computation time for step (6) is proportional to the number of matrices $\mathbf{A}'_{(-d)}$ that can be generated by removing d rows from $\mathbf{A}' \in \mathscr{A}_k$. For a given d, we should choose k^* that minimizes $N_{(-d)}(k)$, namely $k^* = \operatorname{argmin}_k N_{(-d)}(k)$, and then, test the ranks of the resulting reduced matrices. Note that for a given d, the possible value for k is bounded in the range $[|S|/(d-|S|+1), n]$, as a direct result from the conditions specified in Theorem 6 and Corollary 12. If $|S|$ is considerably large, Algorithm 3 may become identical to Algorithm 1, and will not gain any additional computation benefit. When there is no border or a just one-column border in $\mathbf{A}$, $(n/(n-1))|S| - 1 < 1$. As a result, Algorithm 3 will reap the greatest computation benefit by testing those small matrices in $\mathscr{A}_1$ except when $d=1$. For any other cases, the computation saving lies in between.

4.3. Finding girth of a vector matroid

Sections 4.1 and 4.2 provide the procedures of finding the cogirth of a vector matroid. The girth of a vector matroid can be found by solving the cogirth problem of its dual matroid. For a vector matroid, an explicit representation for M^* can be constructed using the standard representative matrix.

Assuming that not all elements in $\mathbf{A}$ are zero, we can change $\mathbf{A}$ into the from $\mathbf{A}_s = (\mathbf{I}_r|\mathbf{D})$ such that $M[\mathbf{A}] = M[\mathbf{A}_s]$, where $r = r(M)$, $\mathbf{I}_r$ is the $r \times r$ identity matrix, and $\mathbf{D}$ is some $r \times (m-r)$ matrix. We call the matrix $\mathbf{A}_s$ a *standard representative matrix* for M. Given the standard representative matrix for M as defined above, the standard representative

$$
\mathbf{A}^T = \begin{pmatrix}
1 & 1.8 & -1.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -0.35 & 0.35 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -0.28571 & 0.28571 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -0.35 & 0.35 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1.09 & -0.09 \\
1 & -0.54 & 0.32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1.09 & -0.09 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.57143 & 0.42857 & 0 & 0 & 0 \\
0 & 1.26 & -0.26 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0.25 & -0.64 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -0.9 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.16 & 1.16 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0.21084 & 0 & 1 & -0.35 & 0.13916 & -0.96755 & 0.238 & 0 & -0.032454 & 0.02 & -0.238 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.28571 & -0.28571 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -0.28571 & 0.28571 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1.2857 & -0.28571 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.12 & -0.56 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1.09 & -0.09 & 0 & 0 & 0 \\
0 & 0.56 & 0.44 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0.55 & -0.55 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.14 & 1.14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.375 & -0.375 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.375 & 1.375 \\
0 & 0 & 0 & 0 & 0.465 & 0.535 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.2857 & -0.28571 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.14286 & 1.1429 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2.25 & -2.25 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2.4 & -1.92 \\
1 & 1.8 & -1.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1.125 & 1.125 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -0.35 & 0.35 \\
-1 & -0.33133 & 0 & 1 & 0.55 & -0.21867 & -1.0041 & -0.02975 & 0 & 0.01 & 0.01 & 0.02975 \\
0 & 0 & 0 & 0 & 1.09 & -0.09 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Fig. 2. 12×34 matrix **A**.

matrix for $M^*[\mathbf{A}]$ is in the form $(-\mathbf{D}^{\mathrm{T}}|\mathbf{I}_{m-r})$. Using the standard representative matrices, the girth problem of a vector matroid can be easily converted to the cogirth problem of the dual matroid or vice versa.

If the field $\mathbb{F}$ is a finite field, e.g., $\mathbb{GF}(2)$, the standard representative matrix is free of round-off errors so that $M[\mathbf{A}]$ is precisely same with $M[\mathbf{A}_s]$. However, in many practical situations when the field is rational $\mathbb{Q}$, the round-off errors may be significant if the numerical routine is not so capable [6]. Fortunately, the widely available methods such as partial or complete pivoting are considered to be capable of generating a standard representative matrix so that $M[\mathbf{A}] = M[\mathbf{A}_s]$ still holds in a practical sense.

4.4. Example

Fig. 2 gives a 12×34 matrix $\mathbf{A}$, of which $R = \mathrm{Row}(\mathbf{A}) = \{1, 2, \ldots, 12\}$ and $E = \mathrm{Col}(\mathbf{A}) = \{1, 2, \ldots, 34\}$. First, we use the n-way hypergraph partition tools to identify the structure in $\mathbf{A}$. We find that the 4-way hypergraph partition (i.e., $n = 4$) gives the minimum of $(n/(n-1))|S| = \frac{4}{3} \times 2$. Hence,

- $S = \{13, 33\}$,
- $E_1 = \{1, 6, 7, 9, 11, 19, 21, 30\}$,
- $E_2 = \{2, 10, 12, 15, 16, 20, 25, 34\}$,
- $E_3 = \{3, 4, 8, 14, 17, 18, 26, 27\}$, and
- $E_4 = \{5, 22, 23, 24, 28, 29, 31, 32\}$.

The permuted $\mathbf{A}$ in a BBDF is shown in Fig. 3. Second, use Algorithm 3 to find the cogirth $g^*(M)$. Given $(n/(n-1))|S| = \frac{8}{3}$, it means the bound on d allowing us to switch to submatrices testing is $\lceil (n/(n-1))|S| \rceil - 1 = 2$ where $\lceil a \rceil$ is the smallest integer greater than or equal to a. Thus, we test $\mathbf{A}$ for $d = 1$, and we conclude that $g^*(M) > 1$. Starting with $d = 2$, we can test submatrices. When $d = 2$, the possible values for k is 2, 3, or 4. We find that $N_{(-2)}(4) = 561$ is the smallest one among the three alternatives. Hence, $k^* = n = 4$ for $d = 2$, and the ranks of $\mathbf{A}'_{(-2)}$ for $\mathbf{A}' \in \mathscr{A}_4$, which

$$
\mathbf{A}^T = \begin{pmatrix}
1 & 1.8 & -1.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -0.54 & 0.32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1.09 & -0.09 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.26 & -0.26 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -0.9 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.56 & 0.44 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.14 & 1.14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1.8 & -1.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -0.35 & 0.35 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0.25 & -0.64 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.16 & 1.16 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -0.28571 & 0.28571 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1.2857 & -0.28571 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0.55 & -0.55 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.465 & 0.535 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.09 & -0.09 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -0.28571 & 0.28571 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -0.35 & 0.35 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.57143 & 0.42857 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.28571 & -0.28571 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.12 & -0.56 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1.09 & -0.09 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.2857 & -0.28571 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.14286 & 1.1429 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1.09 & -0.09 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.375 & -0.375 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.375 & 1.375 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2.25 & -2.25 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2.4 & -1.92 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1.125 & 1.125 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -0.35 & 0.35 \\
-1 & 0.21084 & 0 & 1 & -0.35 & 0.13916 & -0.96755 & 0.238 & 0 & -0.032454 & 0.02 & -0.238 \\
-1 & -0.33133 & 0 & 1 & 0.55 & -0.21867 & -1.0041 & -0.02975 & 0 & 0.01 & 0.01 & 0.02975
\end{pmatrix}
$$

Fig. 3. The permuted **A** in a BBDF.

Table 1
Experimental results by Algorithms 1 and 3

	Size	g^*	$\lvert S\rvert$	n	Time by Algorithm 1	Time by Algorithm 3
Example 1	12×34	7	2	4	203.7 s	0.11 s
Example 2	27×63	8	3	9	148.2 h	7.8 h

are the ranks of $\mathbf{A}_{(-2)}$, are tested. Again, we find that $g^*(M) > 2$. The procedure is repeated for $d = 3, 4, 5, 6$, and 7. In those cases, $k^* = 1$. Eventually, we find that $g^*(M[\mathbf{A}]) = 7$.

The number of tested matrices by Algorithm 3 is 4,243. By comparison, had we used Algorithm 1, the exhaustive rank testing, the number of tested matrices would have been 7,055,731. Algorithm 3, coded in C++ and using singular value decomposition (SVD), takes 0.11 s to find the cogirth on a Pentium 4 HT 3.6 GHz computer while Algorithm 1, also coded in C + + and using SVD, take 203.67 s to finish on the same computer. We tested a larger matrix of dimension 27×63. To save space, we do not include this matrix but its structure parameters are $n = 9$, and $|S| = 3$. With this example, Algorithm 3 takes 7.8 h to find the cogirth $g^* = 8$ while Algorithm 1 finds the solution after 148.2 h. The computational results are summarized in Table 1.

In the above examples, the circuit enumeration algorithm [5] is not included in the comparison because of the following reasons. As we explained earlier, the circuit enumeration is generally wasteful when used to find cogirth since enumerating all the cocircuits is unnecessary and computationally expensive. Another problem is that the circuit enumeration algorithm may be entrapped in an infinite loop when it generates a non-minimal (co)circuit due to numerical round-off errors. This numerical instability gets worse when the matrix size gets large. During our implementation and testing, the circuit enumeration algorithm was not able to complete its procedure for the above two examples due to this numerical instability. We instead tested a smaller problem for a 12×26 matrix. The structure parameters for this matrix are $n = 4$, $|S| = 2$, and $g^* = 5$. It takes 2.4 h to enumerate 4,116 cocircuits. By contrast, Algorithms 1 finds the cogirth in 0.625 s after performing 83,681 rank testings and Algorithm 3 does so in 0.015 s after performing 1,079 rank testings.

5. Concluding remarks

We investigate the (co)girth problem for general matroids and provide theorems characterizing the properties of the (co)girth. Theorems 4 and 6 have been derived based upon matroid duality and connectivity. Lemma 11 and Corollary 12 lead us to developing Algorithms 2 and 3 for vector matroids. To characterize a vector matroid connectivity, we introduce the BBDF of a matrix, and employ the hypergraph partitioning tools to obtain the BBDF. These algorithms could be recursively applied to the vector matroid defined on each sub-matrix when such a sub-matrix is still of large dimension and is embodied with inherent structures.

As we briefly mentioned in Section 1, there are a number of practical applications of the (co)girth-finding algorithms. In a sensor network application, Staroswiecki et al. [13] used a linear system, $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{e}$, to model the source-measurement relationship. They pointed out that the degree of sensor redundancy is a critical quantity to reflect the reliability of a sensor network in the presence of sensor failures. It can be shown that the degree of sensor redundancy is in fact the cogirth of $M[\mathbf{H}^{\mathrm{T}}]$. Another application is related to robust regression estimators. In a linear regression model such as $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$, devising a robust regression estimator needs to know the size of the maximum hyperplane associated with $\mathbf{X}$ [8]. It is not difficult to see that the size of the maximum hyperplane equals $|E| - g^*(M[\mathbf{X}^{\mathrm{T}}])$, where E is the set of row labels of $\mathbf{X}$. Apparently, an efficient cogirth-finding algorithm will benefit these applications significantly.

A final note is that since the algorithms obtain the cogirth of a vector matroid, we can also apply them to the girth and the cogirth problems of representable matroids. However, applications of these theorems to devising (co)girth-finding algorithms in a non-representable matroid may not be straightforward.

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